3.2 – Norm, Dot Product, and Distance in R^n

Definition: If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the **norm** of $\mathbf{v}$ (also called its **length** or **magnitude**) is denoted by $\|\mathbf{v}\|$ [by this author], and is defined by the formula $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$.

Theorem 3.2.1 Properties of the norm of a vector
If $\mathbf{v}$ is a vector in R^n and k is any scalar, then:
a) $\|\mathbf{v}\| \geq 0$
b) $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$
c) $\|k\mathbf{v}\| = |k|\,\|\mathbf{v}\|$ $\rightarrow$ $k\vec{v} = (kv_1, kv_2, \ldots, kv_n) \Rightarrow \|k\vec{v}\| = \sqrt{k^2v_1^2 + k^2v_2^2 + \cdots + k^2v_n^2}$
$= \sqrt{k^2}\,\|\vec{v}\| = |k|\,\|\vec{v}\|$

Definition: The norm of a **unit vector** is 1. We can obtain a unit vector from a nonzero vector $\mathbf{v}$ by multiplying by the reciprocal of its length. This process is called **normalizing** the vector. $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$ has length 1

#1 Find the norm of $\mathbf{v}$ and a unit vector that is oppositely directed to $\mathbf{v}$.
a. $\mathbf{v} = (2,\ 2,\ 2)$
b. $\mathbf{v} = (1,\ 0,\ 2,\ 1,\ 3)$

a. $\|\vec{v}\| = \sqrt{2^2+2^2+2^2} = \sqrt{12} = 2\sqrt{3}$

OR $\vec{v} = 2(1,1,1) \Rightarrow \|\vec{v}\| = 2\sqrt{1^2+1^2+1^2} = 2\sqrt{3}$

$-\frac{\vec{v}}{\|\vec{v}\|} = -\frac{1}{2\sqrt{3}}(2,2,2) = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

b) $\|\vec{v}\| = \sqrt{1^2+2^2+1^2+3^2} = \sqrt{15}$ Likewise, we can find $-\frac{\vec{v}}{\|\vec{v}\|}$.

Definition: The **standard unit vectors in $\mathbf{R}^n$** are the standard basis vectors for R^n, $\mathbf{e}_1 = (1,\ 0,\ 0,\ \ldots,\ 0)$, $\mathbf{e}_2 = (0,\ 1,\ 0,\ \ldots,\ 0)$, $\ldots$, $\mathbf{e}_n = (0,\ 0,\ 0,\ \ldots,\ 1)$.

every vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ in R^n can be expressed as linear combinations of the standard basis vectors:

$\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \cdots + v_n \vec{e}_n$

Definition: If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then we denote the **distance** between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$

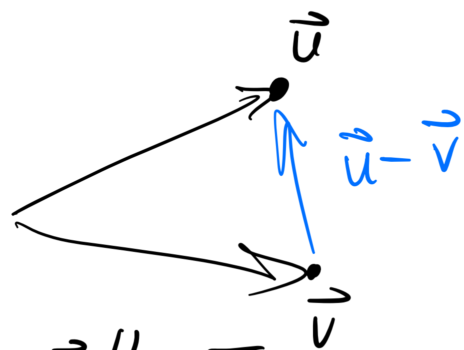
#4 Evaluate the given expression with $\mathbf{u} = (2, -2, 3)$, $\mathbf{v} = (1, -3, 4)$, and $\mathbf{w} = (3, 6, -4)$.

a. $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$

b. $\|\mathbf{u} - \mathbf{v}\|$ ← distance between $\vec{u}$ & $\vec{v}$

c. $\|3\mathbf{v}\| - 3\|\mathbf{v}\|$

d. $\|\mathbf{u}\| - \|\mathbf{v}\|$



a) $\vec{u} + \vec{v} + \vec{w} = (6, 1, 3) \Rightarrow \|\vec{u} + \vec{v} + \vec{w}\| = \sqrt{46}$

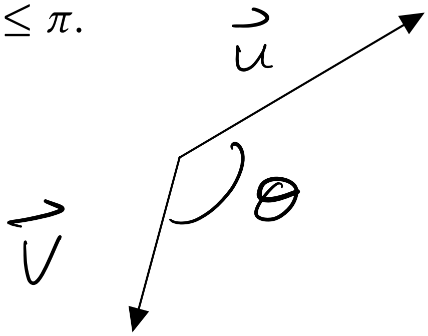
b) $\vec{u} - \vec{v} = (1, 1, -1) \Rightarrow \|\vec{u} - \vec{v}\| = \sqrt{3}$

c) since $\|3\vec{v}\| = 3\|\vec{v}\|$, $\|3\vec{v}\| - 3\|\vec{v}\| = 0$

d) $\|\vec{u}\| - \|\vec{v}\| = \sqrt{17} - \sqrt{26}$

Note that $\|\vec{u} - \vec{v}\| \neq \|\vec{u}\| - \|\vec{v}\|$

Definition: Let **u** and **v** be nonzero vectors in R^n positioned so that their initial points coincide. The **angle between u and v** is the angle θ determined by **u** and **v** such that $0 \leq \theta \leq \pi$.

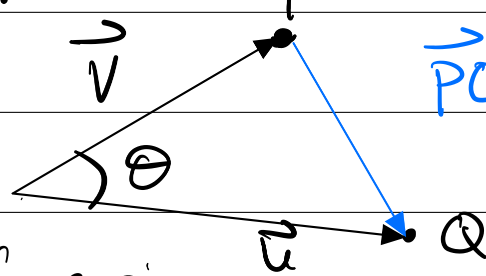


Definition: If **u** and **v** are nonzero vectors in R^2 or R^3, and if θ is the angle between **u** and **v**, then the **dot product** or **Euclidean inner product** of **u** and **v** is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

The dot is not optional.

Definition: If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the **dot product** or **Euclidean inner product** of **u** and **v** is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$.

To connect these two definitions, let $\vec{u}$ & $\vec{v}$ share an initial point in R^3 and let Q & P be their terminal points, respectively.



$\vec{PQ} = \vec{u} - \vec{v}$

By the law of cosines,

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta = \|\vec{PQ}\|^2 = \|\vec{u} - \vec{v}\|^2$$

$$\Rightarrow \|\vec{u}\|\,\|\vec{v}\|\cos\theta = \tfrac{1}{2}\left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2\right)$$

$$\|\vec{u}\|\,\|\vec{v}\|\cos\theta = \frac{1}{2}\Big[\cancel{u_1^2}+u_2^2+u_3^2+\cancel{v_1^2}+v_2^2+v_3^2 - \big((u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2\big)\Big]$$

$$-\cancel{u_1^2}+2u_1v_1-\cancel{v_1^2}+\cdots-$$

$$\Rightarrow \|\vec{u}\|\,\|\vec{v}\|\cos\theta = \boxed{\boxed{u_1v_1+u_2v_2}_{\mathbb{R}^2}+u_3v_3}_{\mathbb{R}^3} = \vec{u}\cdot\vec{v}$$

#10 Find **u** · **v**, **u** · **u**, and **v** · **v**.

a. $\mathbf{u} = (1,\ 1,\ -2,\ 3)$, $\mathbf{v} = (-1,\ 0,\ 5,\ 1)$

b. $\mathbf{u} = (2,\ -1,\ 1,\ 0,\ -2)$, $\mathbf{v} = (1,\ 2,\ 2,\ 2,\ 1)$

b) $\vec{u}\cdot\vec{v} = 2-2+2-2=0$ $\qquad \|\vec{u}\| = \sqrt{10}$

$\vec{u}\cdot\vec{u} = 4+1+1+4 = 10 = \|\vec{u}\|^2$

$\vec{u}=(u_1,u_2,u_3) \Rightarrow \|\vec{u}\| = \sqrt{u_1^2+u_2^2+u_3^2}$; $\vec{u}\cdot\vec{u} = u_1^2+u_2^2+u_3^2$

#11 Find the Euclidean distance between **u** and **v** and the cosine of the angle between those vectors. State whether that angle is acute, obtuse, or 90°.

a. $\mathbf{u} = (3,\ 3,\ 3)$, $\mathbf{v} = (1,\ 0,\ 4)$

b. $\mathbf{u} = (0,\ -2,\ -1,\ 1)$, $\mathbf{v} = (-3,\ 2,\ 4,\ 4)$

a) $d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \sqrt{2^2+3^2+(-1)^2} = \sqrt{14}$

since $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$,

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

b) Find θ. $\quad \vec{u}\cdot\vec{v} = -4-4+4 = -4$

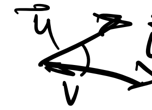
$\|\vec{u}\| = \sqrt{4+1+1} = \sqrt{6}$, $\|\vec{v}\| = \sqrt{9+4+16+16} = \sqrt{45} = 3\sqrt{5}$

$\cos\theta = -\dfrac{4}{3\sqrt{30}}$

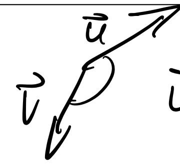
negative $\Rightarrow$ θ is obtuse

$\vec{u}\cdot\vec{v} > 0 \Rightarrow 0 < \theta < 90°$

$\vec{u}\cdot\vec{v} < 0 \Rightarrow 90° < \theta < 180°$



$\vec{u}\cdot\vec{v} > 0$



$\vec{u}\cdot\vec{v} < 0$

Theorem 3.2.2 Properties of the dot product

If **u**, **v**, and **w** are vectors in R^n and if k is a scalar, then:

a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ (symmetry property)

b) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$ (distributive property)

c) $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v}$ (homogeneity property)

d) $\boxed{\mathbf{v}\cdot\mathbf{v} \geq 0}$ and $\boxed{\mathbf{v}\cdot\mathbf{v} = 0 \text{ if and only if } \mathbf{v} = \mathbf{0}}$ (positivity property)

~~Pf~~ (b): Let $\vec{u} = (u_1, u_2, \ldots, u_n)$, $\vec{v} = (v_1, v_2, \ldots, v_n)$, and $\vec{w} = (w_1, w_2, \ldots, w_n) \in R^n$.

def of vector addition

$\vec{u}\cdot(\vec{v}+\vec{w}) = (u_1, u_2, \ldots, u_n)\cdot(v_1+w_1, v_2+w_2, \ldots, v_n+w_n)$

$\sum_{i=1}^{n} u_i(v_i+w_i) = u_1(v_1+w_1) + u_2(v_2+w_2) + \cdots + u_n(v_n+w_n)$ def of dot product

$\sum_{i=1}^{n} (u_iv_i+u_iw_i) = u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \cdots + u_nv_n + u_nw_n$ dist prop

$= u_1v_1 + u_2v_2 + \cdots + u_nv_n + u_1w_1 + u_2w_2 + \cdots + u_nw_n$ comm. prop

$\vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} = u_1v_1 + u_2v_2 + \cdots + u_nv_n + u_1w_1 + u_2w_2 + \cdots + u_nw_n$

def of dot product.

$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

Theorem 3.2.3 More properties of the dot product

If **u**, **v**, and **w** are vectors in R^n and if k is a scalar, then:

a) $\mathbf{0}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{0} = 0$

b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

c) $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{w}$

d) $(\mathbf{u}-\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} - \mathbf{v}\cdot\mathbf{w}$

e) $k(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(k\mathbf{v})$

To prove these, we can use Thm 3.2.2.

Theorem 3.2.4 Cauchy-Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \dots u_n)$ and $\mathbf{v} = (v_1, v_2, \dots v_n)$ are vectors in R^n, then $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$ or in terms of components

$|u_1v_1 + u_2v_2 + \dots + u_nv_n| \leq \left(u_1^2 + u_2^2 + \dots u_n^2\right)^{\frac{1}{2}} \left(v_1^2 + v_2^2 + \dots v_n^2\right)^{\frac{1}{2}}$.

This is a necessary result for the definition of the dot product. Now, $\vec{u}, \vec{v} \in R^n$ gives

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

$0 \leq \cos\theta = \dfrac{|\vec{u} \cdot \vec{v}|}{\|\vec{u}\| \|\vec{v}\|} \leq 1$

Theorem 3.2.5 Triangle Inequalities

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n, then:

a) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (triangle inequality for vectors)

b) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (triangle inequality for distances)

pf: (a) $\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$

$= \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}$

$\leq \|\vec{u}\|^2 + 2|\vec{u} \cdot \vec{v}| + \|\vec{v}\|^2$

(by def of norm and dot product and prop. of abs. value)

$\leq \|\vec{u}\|^2 + 2\|\vec{u}\| \|\vec{v}\| + \|\vec{v}\|^2$

(by the Cauchy-Schwarz inequality)

$= \left(\|\vec{u}\| + \|\vec{v}\|\right)^2$

Taking square roots finishes the proof.

(b) follows from part (a) and def. of distance